# Singapore Mathematical Society

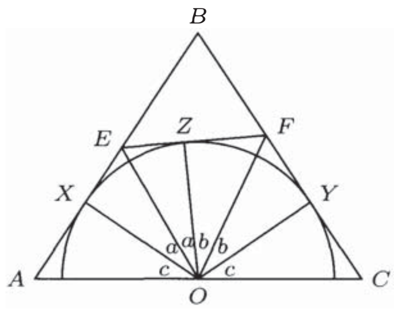
## Singapore Mathematical Olympiad (SMO) 2009

### (Open Section, Round 2 solutions)

**1.** The figure shows half of the rhombus (which is an isosceles triangle), where $X, Y, Z$ are points of tangency of the circle to the sides $AB, CB$ and $EF$ respectively. Note that

$$\angle XOE = \angle EOZ, \quad \angle ZOF = \angle FOY, \quad \angle AOX = \angle COY.$$

In particular, $a + b + c = 90^\circ$.




Thus

$$\angle AEO = 90^\circ - a = b + c = \angle COF.$$

Hence the triangles $AOE$ and $CFO$ are similar. It follows that $AE \cdot CF = AO^2$. Similarly, on the lower half of the rhombus, $AO^2 = AH \cdot CG$. Then $AE/AH = CG/CF$ and hence the triangles $AEH$ and $CGF$ are similar. Thus $\angle AEH = \angle CGF$. Since $AB$ is parallel to $CD$, it follows that $EH$ is parallel to $FG$.

**2.** Let $a_i = 18 + 19i$. We'll show that there are infinitely many $i$ such that $a_i$ consists of only the digit 1, i.e.

$$a_i = 18 + 19i = \frac{10^k - 1}{9}.$$

This yields $10^k \equiv 11 \pmod{19}$. Thus any positive integer of the form $\frac{10^k-1}{9}$, where $10^k \equiv 11 \pmod{19}$ is in the AP. Since $10^6 \equiv 11$ and $10^{18} \equiv 1 \pmod{19}$, we have $10^{18t+6} \equiv 11 \pmod{19}$ for any $t$. Thus there are infinitely many palindromic numbers.

**3.** We have

$$(n+2)A_{n+1} - nA_n = 2(n+1)^{2k}$$
$$(n+1)A_n - (n-1)A_{n-1} = 2(n)^{2k}$$

From these we get

$$\begin{aligned}(n+1)(n+2)A_{n+1} - n(n+1)A_n &= 2(n+1)^{2k+1}\\ n(n+1)A_n - (n-1)nA_{n-1} &= 2(n)^{2k+1}\\ (n+1)(n+2)A_{n+1} - (n-1)nA_{n-1} &= 2(n+1)^{2k+1} + 2(n)^{2k+1}\end{aligned}$$

Using this recurrence, we obtain

$$A_n = \frac{2S(n)}{n(n+1)} \quad \text{where} \quad S(n) = 1^t + 2^t + \cdots + n^t,\ t = 2k+1.$$

Since

$$2S(n) = \sum_{i=0}^{n} \left((n-i)^t + i^t\right) = \sum_{i=1}^{n} \left((n+1-i)^t + i^t\right)$$

we see that $n(n+1) \mid 2S(n)$. Thus $A_n$ is an integer for all $n$.

(i) $n \equiv 1$ or $2 \pmod 4$. Then $S(n)$ is odd since it has an odd number of odd terms. Thus $A_n$ is odd.

(ii) $n \equiv 0 \pmod 4$. Then $(n/2)^t \equiv 0 \pmod n$. Thus

$$S(n) = \sum_{i=0}^{n/2} \left((n-i)^t + i^t\right) - \left(\frac{n}{2}\right)^t \equiv 0 \pmod n.$$

Thus $A_n$ is even.

(iii) $n \equiv 3 \pmod 4$. Then $((n+1)/2)^t \equiv 0 \pmod{n+1}$. Thus

$$S(n) = \sum_{i=1}^{(n+1)/2} \left((n+1-i)^t + i^t\right) - \left(\frac{n+1}{2}\right)^t \equiv 0 \pmod{n+1}.$$

Thus $A_n$ is even.

**4.** First note that

$$\begin{aligned}x_1^3 + x_3^3 + 3x_1x_3 - 1 &= x_1^3 + x_3^3 - (1)^3 - 3x_1x_3(-1)\\ &= (x_1 + x_3 - 1)\left((x_1+x_3)^2 + (x_1+x_3) + 1\right) - 3x_1x_3(x_1+x_3-1)\\ &= (x_1+x_3-1)[(x_1-x_3)^2 + (x_1+1)(x_3+1)].\end{aligned}$$

It is equal to zero only when either $x_1 + x_3 = 1$ or $x_1 = x_3 = -1$. Thus we must have $x_1 + x_3 = 1$ as they are positive. It now suffixes to show that the following is sharp:

$$\sum_{i=1}^{2} \left(y_i + \frac{1}{y_i}\right)^3 \geq 125/4 \text{ when } y_1 + y_2 = 1 \text{ and } y_1, y_2 > 0.$$

To this end, it is clear that the function $f(x) = (x + 1/x)^3$ is convex. Thus,

$$f(x) + f(1 - x) \geq 2f(1/2) = 125/4.$$

**5.** The only solution is $(x, y, z) = (2, 3, 5)$.

First of all, observe that $(x, y) = (x, z) = (y, z) = 1$. Then $2 \leq x < y < z$, and combining the three given congruences we can express it as

$$xy + xz + yz - 1 \equiv 0 \pmod{x,\ y,\ z}.$$

Since $x, y$ and $z$ are pairwise coprime, we have

$$xy + xz + yz - 1 \equiv 0 \pmod{xyz}.$$

It follows that $xy + xz + yz - 1 = k(xyz)$ for some integer $k \geq 1$. Dividing by $xyz$, we obtain that

$$\frac{1}{z} + \frac{1}{y} + \frac{1}{x} = \frac{1}{xyz} + k > 1.$$

Since $x < y < z$, it follows that

$$1 < \frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{3}{x}$$

and this gives $x = 2$ as the only value. In this case, the inequalities give

$$\frac{1}{2} < \frac{1}{y} + \frac{1}{z} < \frac{2}{y},$$

which implies that $y = 3$. It follows that the only possible values of $z$ are 4 and 5. Hence, for $2 \leq x < y < z$, the solutions are $(x, y, z) = (2, 3, 4)$ and $(2, 3, 5)$. Since 2 and 4 are not relatively prime, the only solution is $(x, y, z) = (2, 3, 5)$.