Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

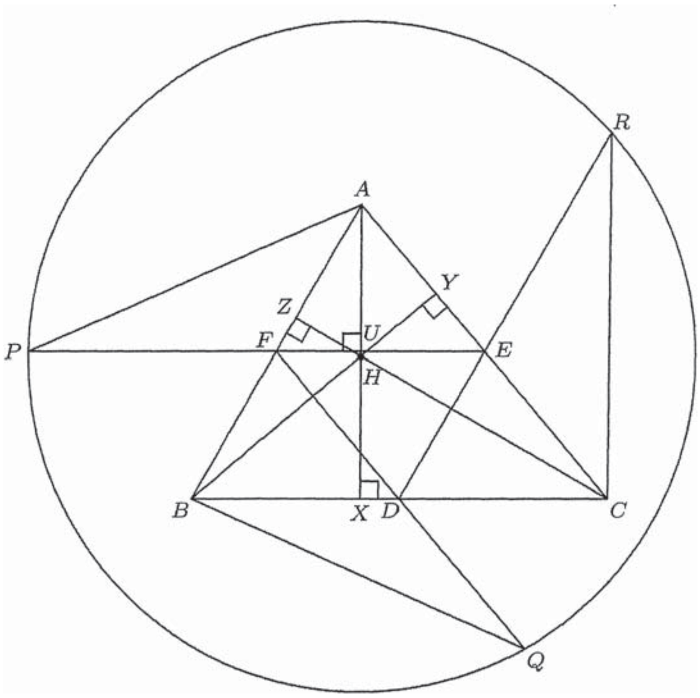
(Open Section, Round 2 solutions)

1. The answer is yes. For any positive integer n, let $f(n)$ be the sum of 2013$^{\text{th}}$ power of the digits of n. Let $S = \{1, 2, \ldots, 10^{2017} - 1\}$, and $n = \overline{a_1 a_2 \ldots a_{2017}} \in S$. Then

$$f(n) = \sum a_i^{2013} \leq 2017 \cdot 9^{2013} < 10^4 \cdot 10^{2013} = 10^{2017} \in S.$$

Since $a_i = f^{(i)}(2013) \in S$, there exist distinct positive integers i, j such that $a_i = a_j$.

2. Let the radius of the circle be r. Let X, Y and Z be the feet of the altitudes from A, B and C respectively. Let PE intersect the altitude from A at U. We have $AP^2 = AU^2 + PU^2 = AU^2 + r^2 - UH^2 = r^2 + (AU + UH) \cdot (AU - UH) = r^2 + AH \cdot (AU - UH) = r^2 + AH \cdot (UX - UH) = r^2 + AH \cdot HX$. Similarly, $BQ = r^2 + BH \cdot HY$, and $CR = r^2 + CH \cdot HZ$. Since $AH \cdot HX = BH \cdot HY = CH \cdot HZ$, we have $AP = BQ = CR$.



3. The result is true for any nonconstant polynomial $f(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_0$ with integer coefficients. We may assume that $a_m > 0$. Thus there exists a positive integer n_0 such that $f(n)$ is positive and increasing on (n_0, ∞).

It suffices to show that if for some $n_1 > n_0$, $f(n_1) = p_1^{r_1} \cdots p_k^{r_k}$ has exactly k distinct prime factors, then for some $n_2 > n_1$, $f(n_2)$ has more than k prime factors. Given such an n_1, let $n_2 = n_1 + p_1^{r_1+1} \cdots p_k^{r_k+1}$. Then

$$f(n_2) \equiv p_1^{r_1} \cdots p_k^{r_k} \pmod{p_1^{r_1+1} \cdots p_k^{r_k+1}}.$$

Hence, for each j, $1 \le j \le k$, we have that $p_j^{r_j}$ divides $f(n_2)$ but $p_j^{r_j+1}$ does not divide $f(n_2)$. As $f(n_2) > f(n_1) = p_1^{r_1} \cdots p_k^{r_k}$, it follows that $f(n_2)$ must have at least $k+1$ prime factors.

4. Because of the second condition above, $0 \notin F$. If F contains only positive elements, let x be the smallest element in F. But then $x = y + z$, and $y, z > 0$ imply that $y, z < x$, a contradiction. Hence F contains negative elements. A similar argument shows that F contains positive elements.

Pick any positive element of F and label it as x_1. Assume that positive elements of F, $x_1, \ldots, x_k$, have been chosen. We can write $x_k = y + z$, where $y, z \in F$. We may assume that $y > 0$. Label y as x_{k+1}. Carry on in this manner to choose positive elements $x_1, x_2, \ldots$ of F, not necessarily distinct. Since F is a finite set, there exist positive integers $i < j$ such that $x_i, \ldots, x_{j-1}$ are distinct and $x_j = x_i$. There are $z_i, \ldots z_{j-1} \in F$ such that

$$\begin{aligned} x_i &= x_{i+1} + z_i \\ x_{i+1} &= x_{i+2} + z_{i+1} \\ \vdots \quad & \quad \vdots \\ x_{j-1} &= x_j + z_{j-1}. \end{aligned}$$

Since $x_i = x_j$, we see that $z_i + z_{i+1} + \cdots + z_{j-1} = 0$. By the assumption, $j - i > n$. Since the elements $x_i, \ldots, x_{j-1}$ are distinct, F contains at least $j - i \ge n + 1$ positive elements. Similarly, F contains at least $n + 1$ negative elements. The result follows.

5. Let the sides be a, b, c. From the sine rule, we have

$$\begin{aligned} \frac{a}{b} &= \frac{\sin 3B}{\sin B} = 4\cos^2 B - 1 \\ \frac{c}{b} &= \frac{\sin C}{\sin B} = \frac{\sin 4B}{\sin B} = 8\cos^3 B - 4\cos B \end{aligned}$$

Thus

$$2\cos B = \frac{a^2 + c^2 - b^2}{ac} \in \mathbb{Q}.$$

Hence there exist coprime positive integers p, q such that $2\cos B = \frac{p}{q}$. Hence

$$\frac{a}{b} = \frac{p^2}{q^2} - 1 \quad \Leftrightarrow \quad \frac{a}{p^2 - q^2} = \frac{b}{q^2};$$
$$\frac{c}{b} = \frac{p^3}{q^3} - \frac{2p}{q} \quad \Leftrightarrow \quad \frac{c}{p^3 - 2pq^2} = \frac{b}{q^3}.$$

Thus

$$\frac{a}{(p^2 - q^2)q} = \frac{b}{q^3} = \frac{c}{p^3 - 2pq^2} = \frac{e}{f}, \qquad \gcd(e, f) = 1.$$

Since perimeter is minimum, $\gcd(a, b, c) = 1$. From $\gcd(e, f) = 1$, we have $f \mid q^3$ and $f \mid p^3 - 2pq^2$. We'll prove that $f = 1$.

If $f > 1$, then it has a prime divisor $f' > 1$ such that $f' \mid q^3$ and $f' \mid p^3 - 2pq^2$. Thus $f' \mid q$ and $f' \mid p$, contradicting $\gcd(p, q) = 1$. Thus $f = 1$. From $\gcd(a, b, c) = 1$, we conclude that $e = 1$. Thus

$$a = (p^2 - q^2)q, \quad b = q^3, \quad c = p^3 - 2pq^2.$$

From $0° < \angle A + \angle B = 4\angle B < 180°$, we get $0° < \angle B < 45°$ and hence $\sqrt{2} < 2\cos B < 2$ implying that $\sqrt{q} < p < 2q$. The smallest positive integers satisfying this inequality is $p = 3, q = 2$. Since $a + b + c = p^2 q + p(p^2 - 2q^2)$ and $p^2 - 2q^2 = 1$, we see that the minimum perimeter is achieved when $p = 3, q = 2$ and the value is 21.