Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 2 Solutions)

1. Without loss of generality, we may assume that all the a_i are positive, else we just change the sign of x_i. Since

$$\left(\frac{\sum_{i=1}^n a_i}{n}\right)^2 \leq \frac{\sum_{i=1}^n a_i^2}{n},$$

we have $\sum_{i=1}^n a_i \leq \sqrt{n}$. There are k^n integer sequences $(t_1, t_2, \ldots, t_n)$ satisfying $0 \leq t_i \leq k-1$ and for each such sequence we have $0 \leq \sum_{i=1}^n a_i t_i \leq (k-1)\sqrt{n}$. Now divide the interval $[0, (k-1)\sqrt{n}]$ into $k^n - 1$ equal parts. By the pigeonhole principle, there must exist 2 nonnegative sequences $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ such that $\left|\sum_{i=1}^n a_i y_i - \sum_{i=1}^n a_i z_i\right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$. Set $x_i = y_i - z_i$ to satisfy the condition.

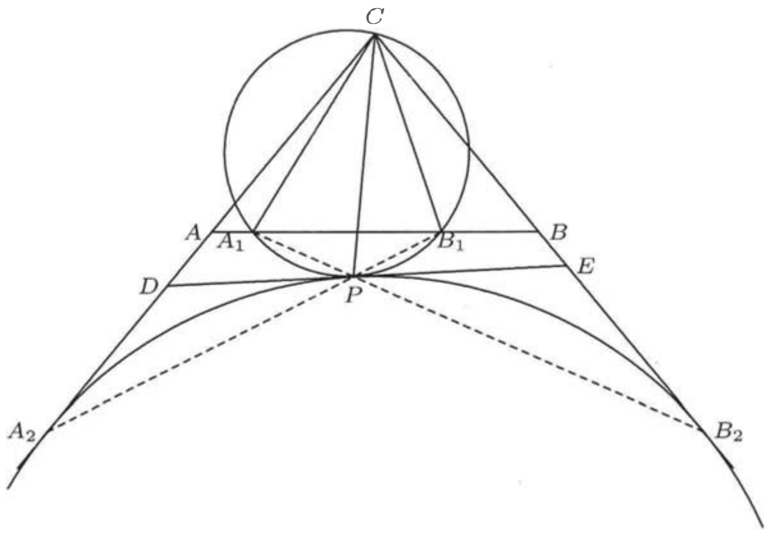
2. Suppose to the contrary that $(x-a_1)(x-a_2)\cdots(x-a_n) - 1 = f(x)g(x)$ for some polynomials $f(x)$ and $g(x)$ with integer coefficients and $\deg(f(x)), \deg(g(x)) \geq 1$. Then $f(a_i)g(a_i) = -1$ for $i = 1, 2, \ldots, n$ implies that $f(a_i) = 1$ and $g(a_i) = -1$ or $f(a_i) = -1$ and $g(a_i) = 1$. Therefore, if we set $h(x) = f(x) + g(x)$, then $h(a_i) = 0$ for all $i = 1, 2, \ldots, n$. As $\deg(h(x)) \leq \max(\deg(f(x)), \deg(g(x))) < n$, the polynomial equation $h(x) = 0$ cannot have n distinct roots. It follows that $h(x)$ must be the zero polynomial. Thus $f(x) = -g(x)$, and therefore

$$(x-a_1)(x-a_2)\ldots(x-a_n) - 1 = -(g(x))^2 \leq 0$$

for all real values of x. But this leads to a contradiction since we can choose a value for x large enough so that $(x-a_1)(x-a_2)\ldots(x-a_n) - 1$ is positive.

2nd solution: We start off as in the first solution. Then instead of defining $h(x)$, we proceed as follows. Let $f(a_i) = 1$, $1 \leq i \leq k$ and $f(a_i) = -1$, $k+1 \leq i \leq n$. Then, $g(a_i) = -1$, $1 \leq i \leq k$ and $g(a_i) = 1$, $k+1 \leq i \leq n$. Therefore $\deg(f(x) - 1) = \deg f(x) \geq \max(k, n-k) \geq \frac{k+(n-k)}{2} = \frac{n}{2}$. Similarly $\deg g(x) \geq \frac{n}{2}$. However $\deg f(x) + \deg g(x) = n$, and thus n is even with $\deg f(x) = \deg g(x) = k = \frac{n}{2}$. Thus $f(x) - 1 = b_1(x-a_1)(x-a_2)\cdots(x-a_k)$, and $g(x)+1 = b_2(x-a_1)(x-a_2)\cdots(x-a_k)$ for some $b_1, b_2 \in \mathbb{Z}$. Together we get $f(x)g(x) + f(x) - g(x) - 1 = b_1 b_2[(x-a_1)(x-a_2)\cdots(x-a_k)]^2$. By comparing coefficient of the x^n term, $b_1 b_2 = 1$. This give us $f(x) - 1 = g(x) + 1$. Similarly, we have $f(x) + 1 = g(x) - 1$, a contradiction.

3. Let the point of contact of the two circles be P. First we show that A_1, P and B_2 are collinear. Let the common tangent at P meet CA at D and CB at E. Let $\angle ABC = b = \angle CAB = \angle A_1CB_1$, $\angle ACB = c$, $\angle A_1CP = x$ and $\angle B_1CP = y$. Then $x + y = b$ and $2b + c = 180°$. We have $\angle PB_1A = x$, $\angle B_1PE = y$. Therefore, by considering PB_1BE, $\angle BEP = 2x$. Hence $\angle EPB_2 = x$ and consequently, $\angle B_1PB_2 = x + y = b$. This implies that A_1, P and B_2 are collinear. Similarly A_2, P and B_1 are collinear. Then $\triangle A_1BC \sim \triangle A_1CB_1$, and $\triangle CAB_1 \sim \triangle A_1CB_1$, whence $\triangle A_1BC \sim \triangle CAB_1$. Thus $AC/AB_1 = A_1B/BC$ and $BC^2 = A_1B \cdot AB_1$, since $AC = BC$. Also $\triangle AA_2B_1 \sim \triangle BA_1B_2$. Thus $AB_1/AA_2 = BB_2/A_1B$ and whence $AA_2^2 = A_1B \cdot AB_1$. Thus B is the midpoint of CB_2. Since $AB \parallel A_2B_2$, we have $A_2B_2 = 2AB$ as required.



2nd solution: Let Γ_1 be the circumcircle of $\triangle A_1B_1C$ and its centre be O_1, let the other circle Γ_2 has center O_2, and the point of contact of the two circles be P. Now since CA_2 and CB_2 are tangent to Γ_2, we have $CA_2 = CB_2$. Together with $CA = CB$, we have $AA_2 = BB_2$. This implies that $\triangle CAB \sim \triangle CA_2B_2$ and $AB \parallel A_2B_2$. Now $\angle A_2O_2B_2 = 180° - \angle ACB = 2\angle CAB = 2\angle A_1CB_1 = \angle A_1O_1B_1$. Thus, isosceles triangles $A_1O_1B_1$ and $A_2O_2B_2$ are similar. Since $AB \parallel A_2B_2$, $A_1O_1 \parallel O_2B_2$, also note that O_1PO_2 is a straight line. Therefore, we have $2\angle A_1B_1P = \angle A_1O_1P = \angle PO_2B_2 = 2\angle PA_2B_2$. This implies that B_1PA_2 is a straight line. Similarly, A_1PB_2 is a straight line. Now we let Γ_1 intersects CA and CB at D and E respectively, and let DA_1 intersects EB_1 at G. By Pascal's Theorem on Γ_1 and the hexagons CEB_1PA_1D, we have A_2, G and B_2 collinear. Using the fact that $\triangle ACB_1 \sim \triangle CA_1B_1 \sim B_1A_1C$, we have $\angle GA_1B_1 = \angle DA_1A = \angle DCB_1 = \angle CA_1B_1$. Similarly, $\angle GB_1A_1 = \angle CB_1A_1$. This implies that G is the image of C under reflection of line AB. Since G is on A_2B_2, A_2B_2 is twice as far as AB from C. Thus, $A_2B_2 = 2AB$.

3rd solution: Let Γ_1 be the circumcircle of $\triangle A_1B_1C$ and its centre be O_1, let the other circle Γ_2 has center O_2. Now since CA_2 and CB_2 are tangent to Γ_2, we have $CA_2 = CB_2$. This implies that $\triangle CAB \sim \triangle CA_2B_2$. Let us perform inversion with center C and radius CA. Let the image of A_1, A_2, B_1, B_2 under this inversion be A_1', A_2', B_1', B_2' respectively. A, B and C remain invariant. The inversion keeps every line that passes through C invariant. Now the image of the line AA_1B_1B is the circumcircle of $\triangle CAB$, let it be Γ_3, and the image of Γ_1 is the line A_1B_1. Thus the image of Γ_2 is tangent to A_1B_1, AC and BC and is thus the incircle of $\triangle ABC$ and touches the sides AC and BC at A_2' and B_2', respectively. Thus $\frac{CA_2'}{CA} = \frac{CB_2'}{CB} = \frac{1}{2}$, which implies that $\frac{CA_2}{CA} = \frac{CB_2}{CB} = 2$. Hence A and B are the midpoints of A_2C and B_2C, respectively. Thus $A_2B_2 = 2AB$.

4. We show that f is the identity function. First we observe that f is an injective function:

$$\begin{aligned} f(m) = f(n) \quad &\Rightarrow \quad f(m) + f(n) = f(n) + f(n) \\ &\Rightarrow \quad f\big(f(m) + f(n)\big) = f\big(f(n) + f(n)\big) \\ &\Rightarrow \quad m + n = n + n \\ &\Rightarrow \quad m = n \end{aligned}$$

Let $k > 1$ be arbitrary. From the original equation, we have the equations

$$f(f(k+1)+f(k-1)) = (k+1)+(k-1) = 2k, \quad \text{and} \quad f(f(k)+f(k)) = k+k = 2k.$$

Since f is injective, we have

$$f(k+1) + f(k-1) = f(k) + f(k) \quad \text{or} \quad f(k+1) - f(k) = f(k) - f(k-1).$$

This characterizes f as an arithmetic progression, so we may write $f(n) = b + (n-1)t$ where $b = f(1)$ and t is the common difference. The original equation becomes $b + [(b + (m-1)t) + (b + (n-1)t) - 1]t = m + n$, which simplifies to $(3b - 2t - 1) + (m+n)t = m + n$. Comparing coefficients, we conclude that $t = 1$ and $b = 1$. Thus $f(n) = n$, as claimed. Clearly, this function satisfies the original functional equation.

5. The answer is $x = 420$.

Let $p_1, p_2, p_3, \ldots$ be all the primes arranged in increasing order. By Bertrand's Postulate, we have $p_i < p_{i+1} < 2p_i$ for all $i \in \mathbb{N}$, thus we have $p_{k+1} < 2p_k < 4p_{k-1} < 8p_{k-2}$ which implies that $64p_kp_{k-1}p_{k-2} > {p_{k+1}}^3$.

Let $p_k \leq \sqrt[3]{x} < p_{k+1}$ for some $k \in \mathbb{N}$. Note that $p_i \mid x$ for $i = 1, 2, \ldots, k$. Suppose $k \geq 5$, then $\sqrt[3]{x} \geq p_5 = 11$. Since $11 > 2^3$ and $11 > 3^2$, we have $2^3 3^2 \mid x$. Since $k \geq 5$, $\gcd(p_kp_{k-1}p_{k-2}, 2^3 3^2) = 1$ and thus $2^3 3^2 p_kp_{k-1}p_{k-2} \mid x$. This means we have $x \geq 72p_kp_{k-1}p_{k-2} > 64p_kp_{k-1}p_{k-2} > {p_{k+1}}^3$, implying $p_{k+1} < \sqrt[3]{x}$, which is a contradiction.Thus $k < 5$ and consequently, $\sqrt[3]{x} < 11$ or $x < 1331$.

Next, we notice that the integer 420 is divisible by all positive integers $\leq \sqrt[3]{420}$, thus $x \geq 420 \Rightarrow \sqrt[3]{x} > 7$. It then follows that x is divisible by $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$.

Finally, suppose $\sqrt[3]{x} \geq 9$. We then have $2^3 \cdot 3^2 \cdot 5 \cdot 7 \mid x$, i.e., $x \geq 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, which is a contradiction since $x < 1331$. Thus $\sqrt[3]{x} < 9$.= or $x < 729$. Since $420 \mid x$ and $x < 729$, we have $x = 420$.

Alternatively, since $x < 1331$ and $420 \mid x$, we only need to check the cases $x = 420, 840, 1260$.