Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 1 Solution)

1. Answer: 12560

$$\sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0 = \frac{2\sin 10^0 \cos 10^0 \cos 20^0 \cos 30^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{\sin 20^0 \cos 20^0 \cos 30^0 \cos 40^0}{2\cos 10^0}$$

$$= \frac{\sin 40^0 \cos 40^0 \cos 30^0}{4\cos 10^0}$$

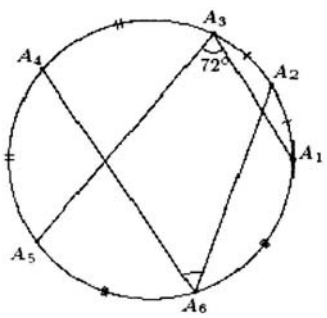
$$= \frac{\sin 80^0 \cos 30^0}{8\cos 10^0}$$

$$= \frac{\cos 10^0 \sin 60^0}{8\cos 10^0}$$

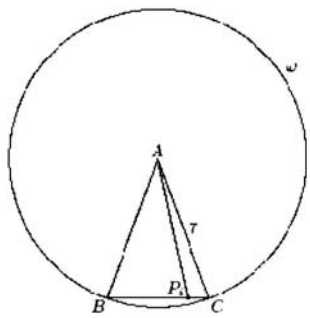
Hence $1000\sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0 = 125\sin 60^0$, showing that $a = 125$ and $b = 60$. So, $100a + b = 12560$.

2. Answer: 54.

First, observe that $\angle A_1A_6A_5 = 180^o - 72^o = 108^o$. Hence $\angle A_4A_6A_2 = \angle A_1A_6A_5 - \angle A_1A_6A_2 - \angle A_4A_6A_5 = 108^o - \angle A_2A_6A_3 - \angle A_3A_6A_4 = 108^o - \angle A_4A_6A_2$. Thus, $2\angle A_4A_6A_2 = 108^o$, resulting in $\angle A_4A_6A_2 = 54^o$.



3. Answer: 2009



Construct a circle ω centred at A with radius AB = AC = 7. The power of P_i with respect to ω is $P_iA^2 - 7^2$, which is also equal to $-BP_i \cdot P_iC$. Thus, $P_iA^2 + BP_i \cdot P_iC = 7^2 = 49$. Therefore, the sum equals 49 x 41 = 2009.

4. Answer: 8

Since $2x^2 + 6x - 56 - (x^2 - 11x + 24) = x^2 + 17x - 80$, the given equation holds if and only if

$$\left(x^2 - 11x + 24\right)\left(2x^2 + 6x - 56\right) \le 0,$$

Since $|a - b| = |a| + |b|$ if and only if $ab \le 0$. The above inequality reduces to

$$(x-3)(x-8)(x-4)(x+7) \le 0.$$

Since

$$\{x : (x-3)(x-8)(x-4)(x+7) \le 0\} = [-7,3] \cup [4,8],$$

we conclude that the largest value of x is 8.

5. Answer: 1742.

We have

$$f(x) = \frac{(x-2005)(x-2006)(x-2007)(x-2008)}{(-1)(-2)(-3)(-4)}(72) + \frac{(x-2004)(x-2006)(x-2007)(x-2008)}{(1)(-1)(-2)(-3)}(-30) + \frac{(x-2004)(x-2005)(x-2007)(x-2008)}{(2)(1)(-1)(-2)}(32) + \frac{(x-2004)(x-2005)(x-2006)(x-2008)}{(3)(2)(1)(-1)}(-24) + \frac{(x-2004)(x-2005)(x-2006)(x-2007)}{(4)(3)(2)(1)}(24) + 7(x-2004)(x-2005)(x-2006)(x-2007)(x-2008),$$

So that f(2009) = 1742.

6. Answer: 2

$$\frac{\sin 80^o}{\sin 20^o}-\frac{\sqrt{3}}{2\sin 80^o}=\frac{\sin 80^o}{\sin 20^o}-\frac{\sin 60^o}{\sin 80^o}$$

$$=\frac{\sin^2 80^o-\sin 20^o\sin 60^o}{\sin 20^o\sin 80^o}=\frac{1-\cos 160^o+\cos 80^o-\cos 40^o}{2\sin 20^o\sin 80^o}$$

$$=\frac{1-\cos 40^o+\cos 80^o-\cos 160^o}{2\sin 20^o\sin 80^o}$$

$$=\frac{2\sin^2 20^o+2\sin 120^o\sin 40^o}{2\sin 20^o\sin 80^o}=\frac{2\sin^2 20^o+2\sqrt{3}\sin 20^o\cos 20^o}{2\sin 20^o\sin 80^o}$$

$$=\frac{\sin 20^o+\sqrt{3}\cos 20^o}{\sin 80^o}=\frac{2\sin(20^o+60^o)}{\sin 80^o}=2$$

7. Answer: 64

Let $N=\overline{abcdefgh}$ be such a number. By deleting a and b, we get $\overline{bcdefgh}$ and $\overline{acdefgh}$ respectively. Both of them are divisible by 7, hence their difference 1000000 ($b-a$) is also divisible by 7, therefore $b-a$ is divisible by 7. By the similar argument, $c-b$, $d-c$, $e-d$, $f-e$, $g-f$, $h-g$ are divisible by 7. In other word, all the digits of N are congruent modulo 7. If N contains digits that are greater than 7, then one can subtract 7 from each digit to get a new number N'. Then N satisfies the requirements in the question if and only if N' does. Since all the digits are congruent modulo 7, it remains to consider numbers of the form $\overline{pppppppp}$, where $p=0, 1, 2, 3, 4, 5, 6$. By deleting a digit in this number, we get the number $\overline{ppppppp}=1111111p$, which is divisible by 7 if and only if p is the digit 0 or 7. However, the first two digits of N must be 7 since the number N has 8 digits and that any number we get by deleting a digit in N has 7 digits. On the other hand, the remaining 6 digits can be independently 0 or 7. Consequently, there are $2^6=64$ choices of such numbers.

8. Answer: 500

$$\sqrt{a} = \sqrt{b} + 20$$
$$a = b + 400 + 40\sqrt{b}$$
$$a - 5b = 400 + 40\sqrt{b} - 4b$$
$$a - 5b = 400 - 4(\sqrt{b} - 5)^2 + 100$$
$$a - 5b \leq 500.$$

9. Answer: 224

Let the circle with centre D and radius r touch the tangent lines AC, BA produced and BC produced at the points E, F and G respectively. Then r = DE = DF = DF. Hence, triangles BDF and BDG are congruent, and hence ∠ABD = ∠CBD = ½ ∠ABC. We have

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{8^2+7^2-9^2}{2(8)(7)} = \frac{2}{7}$, and hence $\sin\frac{B}{2} = \sqrt{\frac{1-\cos B}{2}} = \sqrt{\frac{5}{14}}$.

To find r, we have

$$(ABD) + (BCD) - (ACD) = (ABC),$$

where (ABD) denotes the area of triangle ABD, etc.

Hence $\frac{1}{2}cr + \frac{1}{2}ar - \frac{1}{2}br = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{8+9+7}{2} = 12$.

Solving, we get $r = 4\sqrt{5}$. Considering, triangle BDF, we have $BD = \frac{r}{\sin\frac{B}{2}} = 4\sqrt{14}$. Thus, we have $BD^2 = 224$.

10. Answer: 2009

Since $x = \frac{1}{2}\left(\sqrt[3]{2009} - \frac{1}{\sqrt[3]{2009}}\right)$, we have $(\sqrt[3]{2009})^2 - 2x\sqrt[3]{2009} - 1 = 0$ .We see that $\sqrt[3]{2009}$ is a root of the equation $t^2 - 2xt - 1 = 0$. Thus

$\sqrt[3]{2009} = \frac{2x+\sqrt{4x^2+4}}{2} = x + \sqrt{x^2+1}$ or $\sqrt[3]{2009} = \frac{2x-\sqrt{4x^2+4}}{2} = x - \sqrt{x^2+1} < 0$,

which is not possible. Thus $\left(x + \sqrt{1+x^2}\right)^3 = 2009$.

11. Answer: 90335

There are two cases to consider: Case (1) $x \in \{y, z\}$ and Case (2): $x \notin \{y, z\}$. For Case (1), there are $2\binom{30}{3}$ ways and for Case (2), there are $3\binom{30}{4}$ ways. Hence, total number of ways = 90335.

12. Answer: 38889

Let $S = \{1, 2, 3,\ldots.., 99999\}$, and $S_i = \{n \in S\text{: f}(n) = i\}$ for $i \geq 0$. Thus,

$$S = \cup_{0 \leq i \leq 4} S_i.$$

For $0 \leq i \leq 4$, if $n \in S_i$, and n has exactly k digits in the decimal representation, then exactly we have $k - i$ digits are non-zero. Thus,

$$|S_i| = \sum_{k=i+1}^{5}\binom{k-1}{i}9^{k-i}.$$

Then, it is clear that

$$M = |S_1| + 2|S_2| + 3|S_3| + 4|S_4| = 38889.$$

13. Answer: 96

Note that $3^2 - 1\times1^2 = 8$ and $4^2 - 2\times2^2 = 8$. Suppose $k \equiv 0 \pmod 3$. Note that $a^2 \equiv 0,1 \pmod 3$ for all natural numbers a. Thus, $x^2 - ky^2 \equiv 0,1 - 0 \pmod 3 \equiv 0,1 \pmod 3$ but $8 \equiv 2 \pmod 3$. Hence, there are no natural numbers x and y such that $x^2 - ky^2 = 8$ if k is a multiple of 3. Therefore, $\max\{k\} - \min\{k\} = 99 - 3 = 96$.

14. Answer: 21855.

Let $S_i = \{x \in S\text{: } x \equiv i \pmod 3\}$ for $i = 0,1, 2$. Note that $|S_0| = 5$, $|S_1| = 6$ and $|S_2| = 5$. Let ∂ be the set of all subsets A of S such that $\sum_{x\in\partial} x$ is a multiple of 3. Note that for any $A \subseteq S$,

$$\sum_{x\in A} x = \sum_{i=0}^{2}\sum_{x\in A\cap S_i} x \equiv |A \cap S_1| + 2|A \cap S_2| \pmod 3.$$

Thus, $A \in \partial$ if and only if $|A \cap S_1| \equiv |A \cap S_2| \pmod 3$. Thus, it is clear that

$|\partial| = 2^{|S_0|}m$, where

$$m = \left\{\binom{6}{0}+\binom{6}{3}+\binom{6}{6}\right\}\left\{\binom{5}{0}+\binom{5}{3}\right\}+\left\{\binom{6}{1}+\binom{6}{4}\right\}\left\{\binom{5}{1}+\binom{5}{4}\right\}$$
$$+\left\{\binom{6}{2}+\binom{6}{5}\right\}\left\{\binom{5}{2}+\binom{5}{5}\right\} = 683.$$

Hence $|\partial| = 2^5$ x 683 = 21856. Since we want only non-empty subsets, we have 21855.

15. Answer: 4021

Given $f(x)f(y) = f(2xy + 3) + 3f(x + y) - 3f(x) + 6x$, so if interchanging x and y we have

$f(y)f(x) = f(2xy + 3) + 3f(x + y) - 3f(y) + 6y$.
Subtracting , we have $-3f(x) + 6x = -3f(y) + 6y$ for all $x, y \in \mathbf{R}$, showing that $f(x) - 2x$ is a constant, let it be k. So, $f(x) = 2x + k$.
Substitute back to the given functional equation, we have

$(2x + k)(2y + k) = 2(2xy + 3) + k + 3[2(x + y) + k] - 3(2x + k) + 6x$

$4xy + 2kx + 2ky + k^2 = 4xy + 6 + k + 6x + 6y + 3k - 6x - 3k + 6x$
$2k(x + y) - 6(x + y) = k - k^2 + 6$
$(k - 3)(k + 2) = 2(x + y)(3 - k)$

$(k - 3)(k + 2 + 2x + 2y) = 0$ for all $x, y \in \mathbf{R}$.

Thus $k = 3$. Hence $f(2009) = 2(2009) + 3 = 4021$.

16. Answer: 89970

$a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n = 0 \quad \Rightarrow \quad \dfrac{a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n}{a_{n+1}a_n} = 0$

$\dfrac{a_{n+2}}{a_{n+1}} - \dfrac{a_{n+1}}{a_n} = 1$. From here, we see that $\left\{\dfrac{a_{n+1}}{a_n}\right\}$ is an arithmetic sequence with first term

2009 and common difference 1. Thus $\dfrac{a_{n+1}}{a_n} = n + 2008$, and that

$\dfrac{a_{993}}{a_{992}} = 992 + 2008 = 3000$ and $\dfrac{a_{992}}{a_{991}} = 991 + 2008 = 2999$. We therefore have

$\dfrac{a_{993}}{100a_{991}} = 30(2999) = 89970.$

17. Answer: 6336

Let f_n be the number of ways of tiling a 4 x n rectangle. Also, let g_n be the number of ways of tiling a 4 x n rectangle with the top or bottom two squares in the last column missing, and let h_n be the number of ways of tiling a 4 x n rectangle with the top and bottom squares in the last column missing. Set up a system of recurrence relations involving f_n , g_n and h_n by considering the ways to cover the nth column of a 4 x n rectangle. If two vertical tiles are used, then there are f_{n-1} ways. If one vertical tile and two adjacent horizontal tiles are used, then there are $2g_{n-1}$ ways. If one vertical tile and two non-adjacent horizontal tiles are used, then there are h_{n-1} ways. If four horizontal tiles are used, then there are f_{n-2} ways. Similarly, one can establish the recurrence relations for g_n and h_n. In conclusion, we obtain for $n \geq 2$,

$$
\begin{aligned}
f_n &= f_{n-1} + f_{n-2} + 2g_{n-1} + h_{n-1} \\
g_n &= g_{n-1} + f_{n-1} \\
h_n &= h_{n-2} + f_{n-1},
\end{aligned}
$$

With initial conditions $f_0 = f_1 = g_1 = h_1 = 1$ and $h_0 = 0$. Solving f_n recursively, we obtain $f_9 =$ 6336.

18. Answer: 11439

Each 16-digit binary sequence containing exactly nine '0's and seven '1's can be matched uniquely to such a 7-digit integer or 0000000 as follows: Each '1' will be replaced by a digit from 0 to 9 in this way: the number of '0's to the right of a particular '1' indicates the value of the digit. For example, 0110000010101101 ~ 8832110 and 1111000000000111 ~ 9999000.

Thus, required number = $\binom{16}{7} - 1$.

19. Answer: 127

Since $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + a + 1)$ is a prime, then $a = 2$. Suppose $n = rs$ is a composite, then $2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \cdots + 2^r + 1)$, where each factor on the right is greater than 1, contradicting the fact that $a^n - 1$ is a prime. Therefore n must be a prime. The largest prime n such that $2^n - 1 < 5000$ is 11. However, $2^{11} - 1 = 2047 = 23 \times 89$, which is not a prime. Since $2^7 - 1 = 127$ is a prime number, the answer to this question is 127.

20. Answer: 2008

Let $S = \frac{x_1}{x_1+x_2} + \frac{x_2}{x_2+x_3} + \frac{x_3}{x_3+x_4} + \cdots + \frac{x_{2009}}{x_{2009}+x_1}$. Then it is clear that $S > \frac{x_1}{x_1+x_2+\cdots+x_{2009}} + \frac{x_2}{x_1+x_2+\cdots+x_{2009}} + \cdots + \frac{x_{2009}}{x_1+x_2+\cdots+x_{2009}} = 1$. Next, set

$S' = \frac{x_{2009}}{x_{2009}+x_{2008}} + \frac{x_{2008}}{x_{2008}+x_{2007}} + \frac{x_{2007}}{x_{2007}+x_{2006}} + \cdots + \frac{x_1}{x_1+x_{2009}}$. By the same reasoning, $S' > 1$.

Note that $S + S' = 2009$. We claim that $S < 2008$. Suppose that $S \geq 2008$, then we must have $2009 = S + S' > 2008 + 1 = 2009$, which is absurd. Hence our claim that $S < 2008$ is true.

We shall next show that M is the least possible bound for S. Consider the numbers $x_i = a^i$, where $I = 1, 2, 3, \ldots, 2009$. Direct computation yields $S = \frac{2008}{a+1} + \frac{a^{2008}}{a^{2008}+1}$. When a is chosen to be arbitrarily close to 0, this expression gets arbitrarily close to 2008.

21. Answer: 56

The number of selections such that no two of the numbers are consecutive = The number of binary sequences containing 6 '1's and 34 '0's = $\binom{40}{6}$.

The number of selections such that exactly two of the numbers are consecutive = The number of binary sequences containing a '11', 4 '1's and 35 '0's = $\binom{5}{1}\binom{40}{5}$.

The number of selections with exactly two sets of two consecutive numbers but no three numbers are consecutive = The number of binary sequences containing 2 '11's, 2 '1's and 36 '0's = $\binom{4}{2}\binom{40}{4}$.

The number of selections with exactly three sets of two consecutive numbers but no three numbers are consecutive = The number of binary sequences containing 3 '11's and 37 '0's = $\binom{40}{3}$.

Thus, $1000p = 1000\left[\binom{45}{6} - \binom{40}{6} - \binom{5}{1}\binom{40}{5} - \binom{4}{2}\binom{40}{4} - \binom{40}{3}\right] \div \binom{45}{6} = 56.28..$

22. Answer: 1

$$\sum_{k=0}^{\infty}\frac{2}{\pi}\tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\tan^{-1}\left(\frac{2k+2-2k}{1+(2k)(2k+2)}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{2}{\pi}\left(\tan^{-1}(2k+2)-\tan^{-1}(2k)\right)$$

$$= \lim_{n\to\infty}\frac{2}{\pi}\tan^{-1}(2n+2)=1.$$

23. Answer: 263

We consider the sum $\sum_{k=1}^{11}\left(6k^5+2k^3\right)$ and observe that

$$\sum_{k=1}^{11}\left(6k^5+2k^3\right)$$

$$=\sum_{k=1}^{11}\left\{k^3(k+1)^3-k^3(k-1)^3\right\}$$

$$=11^3\times 12^3.$$

Hence

$$\sum_{k=1}^{11}k^5$$

$$=\frac{11^3\times 12^3}{6}-\frac{2}{6}\sum_{k=1}^{11}k^3$$

$$=2\times 11^3\times 12^2-\frac{1}{3}\left(\frac{11^2\times 12^2}{4}\right)$$

$$=11^2\times 12\times\left(2\times 11\times 12-1\right)$$

$$=11^2\times 12\times 263.$$

Finally, since $\sqrt{256}=16$ and the numbers 2, 3, 5, 7, 11 and 13 do not divide 263, we conclude that 263 is the largest prime factor of $\sum_{k=1}^{11}k^5$.

24. Answer: 15

For each positive integer *n,* we have

$$x_{n+2} - x_{n+1} = -\frac{3}{4}(x_{n+1} - x_n)$$

and so we have $x_{n+1} - x_n = \left(-\frac{3}{4}\right)^{n-1}(x_2 - x_1) = 21\left(-\frac{3}{4}\right)^{n-1}$.

Therefore, $x_n = x_1 + 21\sum_{k=1}^{n-1}\left(-\frac{3}{4}\right)^{k-1}$ for all positive integers $n \geq 3$. --- (1)

Now, we let $n \to \infty$ in (1) to conclude that $\lim_{n\to\infty} x_n = x_1 + 21\sum_{k=1}^{\infty}\left(-\frac{3}{4}\right)^{k-1} = 3 + \frac{21}{1+0.75} = 15.$

25. Answer: 14413

The number of rectangles (including squares)

= (20 + 19 + 18 +…+2 + 1)(30 + 29 + 28 +…..+1) x 2 – $(20 + 19 + 18 +\ldots + 1)^2$

= 151200.

The number of squares

= (20 x 30 + 19 x 29 + 18 x 28 + ….. + 1 x 11) x 2 – $(20^2 + 19^2 + 18^2 + \ldots + 1^2)$

= 7070.

N = the number of rectangles less all squares = 151200 – 7070 = 144130.

Hence *N* / 10 = 14413.